Calculation of the renormalised quantum stress tensor by adiabatic regularisation in two- and four-dimensional Robertson-Walker space-times

1978 J. Phys. A: Math. Gen. 11 603

(http://iopscience.iop.org/0305-4470/11/3/021)

J. Phys. A: Math. Gen., Vol. 11, No. 3, 1978. Printed in Great Britain. © 1978

Calculation of the renormalised quantum stress tensor by adiabatic regularisation in two- and four-dimensional Robertson–Walker space–times

T S Bunch
Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

Received 3 October 1977

Abstract. Adiabatic regularisation is used to obtain the vacuum expectation value of the stress tensor for a conformally coupled massless scalar field in two- and four-dimensional Robertson–Walker space–times. The results obtained agree with previous work using point-splitting (including the accepted value for the anomalous trace) but the details of the calculation are very much simpler and more elegant.

1. Introduction

The problem of obtaining a finite renormalised value for the stress tensor of a quantum field propagating in a curved space–time as an expectation value of some quantum state has generated much recent interest. The main difficulty encountered is that the expressions obtained are always formally infinite and some method of regularisation is required to remove the infinities. One method which has enabled a number of explicit calculations to be performed is known as covariant geodesic point-splitting (Davies and Fulling 1977, Davies *et al* 1977). Early point-splitting calculations led to expressions which were ambiguous because they depended on the tangent vector, t^{μ}, to the geodesic along which points were separated. More recently a method for removing these ambiguities has been developed based on the De Witt–Schwinger expansion, and calculations have been performed which lead to an unambiguous, finite, conserved stress tensor (Bunch and Davies 1978a, b). The procedure is to subtract the terms in the DeWitt–Schwinger expansion calculated by Christensen (1976a, b) which contain no more than n derivatives of the metric, where n is the dimension of the space–time, from the exact expression for $\langle T_{\mu\nu}\rangle$, the expectation value of the stress tensor, obtained by point-splitting. This regularisation procedure is the same as the adiabatic regularisation scheme of Parker and Fulling (Parker and Fulling 1974, Fulling *et al* 1974) a fact which has been verified explicitly in two dimensions by a point-splitting calculation of the divergent integrals obtained in adiabatic regularisation (Bunch 1977, T S Bunch, S M Christensen and S A Fulling 1978, in preparation), giving expressions which agree exactly with those obtained by Christensen using the De Witt–Schwinger expansion.

In this paper it will be shown how two results previously obtained by long point-splitting calculations can be obtained very quickly by using adiabatic regularisation directly. This is achieved by performing the subtractions which affect the regularisation before the integrals are evaluated. The integrals are consequently finite, and point-splitting is therefore not required for their evaluation.

2. Adiabatic regularisation in two dimensions

In this section the renormalised stress tensor for a massless scalar field propagating in a two-dimensional Robertson–Walker space–time will be calculated. The metric is taken in conformally flat form as:

$$ds^2 = C(\eta)(d\eta^2 - dx^2) = C\, du\, dv \tag{2.1}$$

where the null coordinates u and v are $u = \eta - x$, $v = \eta + x$. The normalised positive-frequency mode functions are ϕ_k given by

$$\frac{e^{-iku}}{(4\pi k)^{1/2}} \quad \text{and} \quad \frac{e^{-ikv}}{(4\pi k)^{1/2}}. \tag{2.2}$$

The stress tensor for a massive scalar field is:

$$T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma} (\partial_\rho \phi)(\partial_\sigma \phi) + \tfrac{1}{2} g_{\mu\nu} m^2 \phi^2. \tag{2.3}$$

Thus the components of the stress tensor in the null coordinate system are:

$$T_{uu} = \left(\frac{\partial \phi}{\partial u}\right)^2; \qquad T_{vv} = \left(\frac{\partial \phi}{\partial v}\right)^2; \qquad T_{uv} = \frac{Cm^2}{4} \phi^2. \tag{2.4}$$

The divergent vacuum expectation values of (2.4) for a massless field are, using (2.2):

$$\begin{aligned} \langle T_{uu} \rangle_{\text{div}} &= \int_0^\infty \frac{\partial \phi_k}{\partial u} \frac{\partial \phi_k^*}{\partial u}\, dk = \frac{1}{4\pi} \int_0^\infty k\, dk \\ \langle T_{vv} \rangle_{\text{div}} &= \langle T_{uu} \rangle_{\text{div}}; \qquad \langle T_{uv} \rangle_{\text{div}} = 0. \end{aligned} \tag{2.5}$$

From these must be subtracted the divergent integrals which arise in adiabatic regularisation. These are obtained by taking the general solution of the massive scalar wave equation to be

$$\phi(\eta, x) = (2\pi)^{-1/2} \int dk (A_k \psi_k(\eta) e^{ikx} + A_k^\dagger \psi_k^*(\eta) e^{-ikx}) \tag{2.6}$$

where the functions ψ_k satisfy

$$\frac{d^2 \psi_k}{d\eta^2} + w_k^2 \psi_k = 0 \tag{2.7}$$

with $w_k^2 = k^2 + Cm^2$, and are normalised according to

$$\psi_k \partial_\eta \psi_k^* - \psi_k^* \partial_\eta \psi_k = i. \tag{2.8}$$

A generalised WKB solution of (2.7) has been given by Chakraborty (1973): it is this solution which gives rise to the adiabatic vacuum, $|0\rangle$, with respect to which one obtains $\langle 0|T_{\mu\nu}|0\rangle$, the divergent quantity to be subtracted from (2.5). It is essential to calculate $\langle 0|T_{\mu\nu}|0\rangle$ only up to adiabatic order T^{-n} (n derivatives of the metric) where n

is the dimension of the space–time. This gives for the normalised positive-frequency WKB solutions:

$$\psi_k = (2W_k)^{-1/2} \exp\left(-\mathrm{i}\int^{\eta} W_k(\eta')\,\mathrm{d}\eta'\right) \tag{2.9}$$

where

$$W_k = w_k(1+\epsilon_2)^{1/2} \approx w_k(1+\tfrac{1}{2}\epsilon_2) \tag{2.10}$$

and

$$\epsilon_2 = -w_k^{-3/2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(w_k^{-1}\frac{\mathrm{d}}{\mathrm{d}\eta}w_k^{1/2}\right). \tag{2.11}$$

Substituting (2.6) and (2.9) into the vacuum expectation values of (2.4) gives

$$\langle 0|T_{uu}|0\rangle = \frac{1}{16\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{W_k}\left[(W_k+k)^2+\frac{1}{4}\left(\frac{W_k'}{W_k}\right)^2\right] \tag{2.12}$$

$$\langle 0|T_{vv}|0\rangle = \frac{1}{16\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{W_k}\left[(W_k-k)^2+\frac{1}{4}\left(\frac{W_k'}{W_k}\right)^2\right] \tag{2.13}$$

$$\langle 0|T_{uv}|0\rangle = \frac{Cm^2}{16\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{W_k} \tag{2.14}$$

where W_k is obtained to adiabatic order T^{-2} from (2.10) and (2.11):

$$W_k = w_k + \frac{A^2}{w_k^3} + \frac{B^2}{w_k^5}; \qquad A^2 = -\tfrac{1}{8}C''m^2; \qquad B^2 = \tfrac{5}{32}C'^2m^4. \tag{2.15}$$

The prime denotes differentiation with respect to η. Since $W_k = W_{-k}$, (2.12)–(2.14) can be rewritten as:

$$\langle 0|T_{uu}|0\rangle = \langle 0|T_{vv}|0\rangle = \frac{1}{8\pi}\int_0^{\infty}\frac{\mathrm{d}k}{W_k}\left[W_k^2+k^2+\frac{1}{4}\left(\frac{W_k'}{W_k}\right)^2\right] \tag{2.16}$$

$$\langle 0|T_{uv}|0\rangle = \frac{Cm^2}{8\pi}\int_0^{\infty}\frac{\mathrm{d}k}{W_k}. \tag{2.17}$$

Substitute (2.15) into (2.16) and subtract this from (2.5) to obtain

$$\langle T_{uu}\rangle_{\mathrm{ren}} = -\frac{1}{8\pi}\int_0^{\infty}\frac{(w_k-k)^2}{w_k}\,\mathrm{d}k + \frac{1}{64\pi}\int_0^{\infty}\left(\frac{C''m^2}{w_k^3} - \frac{C''k^2m^2}{w_k^5} - \frac{7}{4}\frac{C'^2m^4}{w_k^5} + \frac{5}{4}\frac{C'^2k^2m^4}{w_k^7}\right)\mathrm{d}k \tag{2.18}$$

where it is understood that the limit $m \to 0$ is to be taken at the end. The first integral is a finite quantity which vanishes as $m \to 0$. The second is independent of m and can be evaluated by the substitution $k = m\sqrt{C}\tan\theta$ to give:

$$\langle T_{uu}\rangle_{\mathrm{ren}} = \frac{1}{96\pi}\left(\frac{C''}{C} - \frac{3}{2}\frac{C'^2}{C^2}\right). \tag{2.19}$$

Similarly

$$\langle T_{uv}\rangle_{\mathrm{ren}} = -\frac{CC''m^4}{64\pi}\int_0^{\infty}\frac{\mathrm{d}k}{w_k^5} + \frac{5CC'^2m^6}{256\pi}\int_0^{\infty}\frac{\mathrm{d}k}{w_k^7} = -\frac{1}{96\pi}\left(\frac{C''}{C} - \frac{C'^2}{C^2}\right) = -\frac{R}{48\pi}g_{uv} \tag{2.20}$$

where R is the Ricci scalar. Consequently one obtains from (2.19) and (2.20)

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = -\frac{R}{48\pi} g_{\mu\nu} + \theta_{\mu\nu}$$

where

$$\theta_{uu} = \theta_{vv} = \frac{1}{96\pi}\left(\frac{C''}{C} - \frac{3}{2}\frac{C'^2}{C^2}\right); \qquad \theta_{uv} = 0$$

which is the result obtained by Davies *et al* (1976).

3. Adiabatic regularisation in four dimensions

In this section the renormalised stress tensor will be obtained for a conformally coupled massless scalar field propagating in a spatially flat four-dimensional Robertson–Walker metric:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{3.1}$$

which implies that

$$ds^2 = C(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{3.2}$$

where $C(\eta) = a^2(t)$ and $\eta = \int a^{-1}\,dt$.

Because of the conformal invariance, positive-frequency-mode solutions exist given by $C^{-1/2}$ times the Minkowski-space-mode solutions. These give rise to $\langle T_{tt}\rangle_{\text{div}}$ given by equation (5.29) of Fulling *et al* (1974):

$$\langle T_{tt}\rangle_{\text{div}} = \frac{1}{4\pi^2 a^4}\int q^2 \Omega_q \, dq \tag{3.3}$$

where $\Omega_q = a\omega_q = (q^2 + m^2 a^2)^{1/2} = (q^2 + Cm^2)^{1/2}$(note that the factor $(4\pi^2 a^2)^{-1}$ in (5.29) of Fulling *et al* should read $(4\pi^2 a^3)^{-1}$). The quantity to be subtracted from this is given by (5.30) of Fulling *et al*:

$$\langle 0|T_{tt}|0\rangle = \frac{1}{8\pi^2 a^4}\int \frac{q^2}{\Omega_q}\left[2\Omega_q^2 + \frac{1}{4}\left(\frac{\Omega_q'}{\Omega_q}\right)^2 - \frac{1}{8}\left(\frac{\Omega_q'}{\Omega_q}\right)^2 \epsilon_2 + \frac{1}{4}\left(\frac{\Omega_q'}{\Omega_q}\right)\epsilon_2' + \frac{1}{4}\Omega_q^2\epsilon_2^2\right] dq \tag{3.4}$$

where the prime denotes differentiation with respect to η. Subtracting (3.4) from (3.3) gives the finite quantity:

$$\langle T_{tt}\rangle_{\text{ren}} = -\frac{1}{8\pi^2 a^4}\int_0^\infty \frac{q^2}{\Omega_q}\left[\frac{1}{4}\left(\frac{\Omega_q'}{\Omega_q}\right)^2 - \frac{1}{8}\left(\frac{\Omega_q'}{\Omega_q}\right)^2 \epsilon_2 + \frac{1}{4}\left(\frac{\Omega_q'}{\Omega_q}\right)\epsilon_2' + \frac{1}{4}\Omega_q^2\epsilon_2^2\right] dq. \tag{3.5}$$

The quantity ϵ_2 is:

$$\epsilon_2 = -\Omega_q^{-3/2}\frac{d}{d\eta}\left(\Omega_q^{-1}\frac{d}{d\eta}(\Omega_q^{1/2})\right) = -\frac{1}{2}\Omega_q''\Omega_q^{-3} + \frac{3}{4}(\Omega_q')^2\Omega_q^{-4}. \tag{3.6}$$

Substituting (3.6) into (3.5) leads to

$$\langle T_{tt}\rangle_{\text{ren}} = -\frac{1}{8\pi^2 a^4}\int_0^\infty q^2[\tfrac{1}{4}(\Omega_q')^2\Omega_q^{-3} - \tfrac{45}{64}(\Omega_q')^4\Omega_q^{-7}$$
$$+ \tfrac{5}{8}\Omega_q''(\Omega_q')^2\Omega_q^{-6} + \tfrac{1}{16}(\Omega_q'')^2\Omega_q^{-5} - \tfrac{1}{8}\Omega_q'''\Omega_q'\Omega_q^{-5}]\, dq. \tag{3.7}$$

Now substitute $\Omega_q = (q^2 + Cm^2)^{1/2}$, integrate, and take the limit $m \to 0$ to obtain:

$$\langle T_{tt} \rangle_{\text{ren}} = -\frac{1}{8\pi^2 a^4}\left(-\frac{1}{240}\frac{C'''C'}{C^2} + \frac{1}{480}\frac{C''^2}{C^2} + \frac{1}{120}\frac{C''C'^2}{C^3} - \frac{1}{192}\frac{C'^4}{C^4}\right).$$

Putting $a^2 = C$, $D = C'/C$ and changing to $\langle T_{\eta\eta} \rangle_{\text{ren}}$ gives:

$$\langle T_{\eta\eta} \rangle_{\text{ren}} = (2880\pi^2 C)^{-1}(\tfrac{3}{2}D''D - \tfrac{3}{4}D'^2 - \tfrac{3}{8}D^4). \tag{3.8}$$

In principle $\langle T_{xx} \rangle_{\text{ren}}$ could be obtained in the same way (although not by using the expressions in Fulling *et al* since these were obtained from $\langle T_{tt} \rangle_{\text{ren}}$ by the assumption, now known to be false, that $\langle T_\alpha{}^\alpha \rangle_{\text{ren}} = 0$). However, since $\langle T_{\mu\nu} \rangle_{\text{ren}}$ must be covariantly conserved, it is much simpler to use the covariant conservation law to fix $\langle T_{xx} \rangle_{\text{ren}}$; this requires that:

$$\partial_\eta \langle T_{\eta\eta} \rangle_{\text{ren}} + \tfrac{1}{2}D\langle T_{\eta\eta} \rangle_{\text{ren}} + \tfrac{3}{2}D\langle T_{xx} \rangle_{\text{ren}} = 0. \tag{3.9}$$

Thus from (3.8) and (3.9) one obtains:

$$\langle T_{xx} \rangle_{\text{ren}} = (2880\pi^2 C)^{-1}(-D''' + \tfrac{1}{2}D''D - \tfrac{1}{4}D'^2 + D'D^2 - \tfrac{1}{8}D^4). \tag{3.10}$$

Finally (3.8) and (3.10) can be combined to give the result obtained by Davies *et al* (1977):

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = (2880\pi^2)^{-1}(-\tfrac{1}{6}{}^{(1)}H_{\mu\nu} + {}^{(3)}H_{\mu\nu})$$

where

$$^{(1)}H_{\mu\nu} = 2R_{;\mu\nu} - 2\,\Box\, Rg_{\mu\nu} + 2(RR_{\mu\nu} - \tfrac{1}{4}R^2 g_{\mu\nu})$$

$$^{(3)}H_{\mu\nu} = -R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \tfrac{1}{12}R^2 g_{\mu\nu}.$$

References

Bunch T S 1977 *PhD Thesis* King's College, London
Bunch T S and Davies P C W 1978a *Proc. R. Soc.* in the press
—— 1978b *J. Phys. A: Math. Gen.* submitted for publication
Chakraborty B 1973 *J. Math. Phys.* **14** 188
Christensen S M 1976a *PhD Thesis* University of Texas at Austin
—— 1976b *Phys. Rev.* D **14** 2490
Davies P C W and Fulling S A 1977 *Proc. R. Soc.* A **354** 59
Davies P C W, Fulling S A, Christensen S M and Bunch T S 1977 *Ann. Phys., NY* in the press
Davies P C W, Fulling S A and Unruh W G 1976 *Phys. Rev.* D **13** 2720
Fulling S A, Parker L and Hu B L 1974 *Phys. Rev.* D **10** 3905
Parker L and Fulling S A 1974 *Phys. Rev.* D **9** 341